

# Shape fluctuations of a Deformable Body in a Randomly Stirred Host Fluid

**Gady Frenkel, Moshe Schwartz**

*Raymond and Beverly Sackler Faculty of Exact Sciences*

*School of Physics and Astronomy*

*Tel Aviv University, Ramat Aviv, 69978, Israel*

Consider a deformable body immersed in an incompressible fluid that is randomly stirred. Sticking to physical situations in which the body departs only slightly from its spherical shape, we investigate the deformations of the body. The shape is decomposed into spherical harmonic modes. We study the correlations of these modes for a general class of random flows that include the flow due to thermal agitation. Our results are general in the sense that they are applicable to any body that is described solely by the shape of its surface.

## I. INTRODUCTION

The current paper is the second in a series of papers that treat systems of deformable objects immersed in an incompressible liquid. These systems are common in nature ( milk, emulsions, composite fluids )and have been studied extensively. Nevertheless, there are many questions concerning them that need to be answered still. This is so, because the actual solution of such systems is extremely difficult. The objects interact with themselves and with each other via the hydrodynamic interactions, and form, in general, a non-linear many-body problem. Moreover, each object has infinite degrees of freedom that correspond to deformations, and all of them , of all the objects, interact. The problem is a little simplified if the deformation of the objects from spherical shape remains small and the fluid is in the linear regime, as the case of the Stokes approximation to the Navier - Stokes equation.

Our final goal is to obtain the response of the composite system to a given velocity field imposed on the liquid. The velocity field we have in mind may be fixed in time like simple shear or randomly fluctuating in time and space. Even in the first case the velocity field, each object experience must have a random part due to the random passage of other objects nearby. Therefore, effects of randomness are important to the understanding of such systems. We wish to know first, what is the response of a single deformable object to the random velocity field, created by the other objects and the external source. Once the response of a single object is known, we can obtain the response of the full, many body, system by using the response of each body as a source of an additional velocity field. In the former paper [1] we studied the motion of the center of a deformable object and derived the Mean Squared Displacement of its center. In this paper we investigate the deformation degrees of freedom, within the scope of the simplifications stated above, thus completing the description of the response of a single object to a random velocity field. Decomposing the deformation into spherical harmonic modes, we consider the correlations between deformation modes. We find, among other things that different modes are decoupled and that the correlation function does not depend on the parameter  $m$  of the  $Y_{l,m}$  spherical harmonics (following from spherical symmetry). We also obtain a method to calculate these correlations as a function of time, build typical drop shapes from the correlation functions, and discuss several interesting cases as thermal agitation.

The plan of this paper is as follows: In section II we describe the system we have in mind and formulate the basic equations. In section III we introduce the deformation coefficients and obtain their correlations for the simple case where the external velocity is uncorrelated in time. General correlation functions are considered in section IV. We obtain the deformation correlations and make simplifications for equal times. In section V we discuss some properties of the correlation function of the external velocity for several cases, among them is the special case of thermal agitation. An algorithm for numerical computation of the correlations in the general case is described in the Appendix.

## II. THE SYSTEM

Consider a single deformable body immersed in a host fluid.

1. The deformable body is fluid, in the sense that the velocity fields are well defined everywhere (both inside and outside the body). In particular, each surface element moves with the velocity of the flow at its position:

$$\dot{\vec{r}} = \vec{v}(\vec{r}) \tag{1}$$

2. Both the body and the host fluid are incompressible,  $\vec{\nabla} \cdot \vec{v} = 0$ .
3. The body is characterized by an energy that depends on its shape ( Changing the orientation or switching places of two surface particles while keeping the shape constant does not cause any energy). The shape of minimum energy is a sphere. The energy may be surface tension [6], Helfrich bending energy [7,8], etc. Deformation of the shape change the energy, exerts force density in the liquid and therefore creates an additional velocity field, denoted  $\vec{v}_\psi$ .
4. We investigate the regime where the hydrodynamic equations are linear in the velocity ( i.e. a velocity field induced by several sources is equal to the linear sum of the velocity fields that each source induces separately). For instance, if the flow is governed by the Navier Stokes equation, then our assumption implies that the Reynolds number is small and that the stokes approximation is applicable. The linearity implies that in our system the actual velocity field is the linear sum of the imposed velocity field ,  $\vec{v}_{ext}$  ( where  $\vec{v}_{ext}$  is the velocity field that would have been the actual velocity field if the body was absent), and the velocity field induced by the deformations,  $\vec{v}_\psi$ ,

$$\vec{v} = \vec{v}_{ext} + \vec{v}_\psi. \tag{2}$$

5. We consider cases in which the external velocity field is random, with an average of zero. The velocity correlation function is known and depends only on distance and

time difference. We also assume that the external velocity is weak enough to allow the body to remain almost spherical

### III. WHITE NOISE FLOW

Consider a spherical body which is slightly deformed. The equation

$$\frac{\rho}{R} + f(\Omega, t) - 1 = 0 \quad (3)$$

defines its surface, providing for each spatial direction,  $\Omega$ , the distance,  $\rho \equiv |\vec{r} - \vec{r}_0|$ , of the surface from the center of the body,  $\vec{r}_0$ .  $R$  is the radius of the undeformed sphere.  $f(\Omega, t)$  that parameterize the shape and is named the deformation function. The deformation function is decomposed into spherical harmonics,  $f(\Omega, t) = \sum_{l=-\infty}^{\infty} \sum_{m=-l}^l f_{lm}(t) Y_{lm}(\Omega)$ , where the deformation coefficients are denoted  $f_{lm}(t)$ . Our goal is to obtain the correlations between deformation coefficients.

The center of the body,  $\vec{r}_0$ , is chosen to be the point around which the deformation coefficients with  $l = 1$  vanish:  $f_{1,m} = 0$ . A different definition of the center will introduce three additional equations for the deformation coefficients with  $l = 1$ . We are not interested in those since in the first order the spherical harmonics with  $l = 1$  describe a translation of the body. In previous papers [1,2] we have used the same definitions to describe the motion of the center. Therefore, their results are consistent with the derivation done here and can be used. Let  $\psi(\vec{r})$  be a scalar field, defined everywhere in such a way that the equation  $\psi(\vec{r}) = 0$  describes the surface of the body [3,4]. Straight forward manipulation of eq. (1) give rise to a continuity equation for  $\psi$ , presented here in a coordinate system that moves with the center of the body:

$$\dot{\psi} + \vec{v}_{\psi} \cdot \vec{\nabla} \psi = -(\vec{v}_{ext} - \dot{\vec{r}}_0) \cdot \vec{\nabla} \psi \quad (4)$$

A good candidate for  $\psi$  is found in equation (3),  $\psi = \frac{\rho}{R} + f(\Omega, t) - 1$ . Assuming that  $|\vec{v}_{ext} - \dot{\vec{r}}|$  is small, the right hand side of eq. (4) is, in the first order, equal to  $Q \equiv \frac{1}{R} \left\{ \hat{\rho} \cdot [\vec{v}_{ext} - \dot{\vec{r}}] \right\}$  (see [3]), where  $\hat{\rho}$  is a unit vector directed outwards from the center of the body. Since

the minimum energy of the body is obtained for a spherical shape, the velocity induced by the body is zero when the sphere is undeformed. Therefore, the velocity  $\vec{v}_\psi$  must be, in general, a linear functional of the deformation  $f(\Omega, t)$  ( with corrections of higher order that we neglect). The term  $\vec{v}_\psi \cdot \vec{\nabla}\psi$  is obtained in the leading order by taking  $\vec{\nabla}\psi$  on the original sphere and  $\vec{v}_\psi$  to first order in the deformation. We are interested in the first order evolution of the surface, and the terms above are already linear in the deformation; Thus, we can calculate them on the original sphere, where it is possible to develop eq. (4) in terms of spherical harmonics (for further explanation see [1]). Consequently, the generic eq. for the deformation coefficient,  $f_{l,m}$ , must be of the form

$$\frac{\partial f_{lm}(t)}{\partial t} + \lambda_l f_{lm}(t) = -Q_{lm}(t), \quad (5)$$

where  $Q_{lm}$  is given by

$$Q_{lm} = \frac{1}{R} \int d\Omega \left\{ \hat{\rho} \cdot [\vec{v}_{ext} - \dot{\vec{r}}_0] Y_{l,m}^*(\Omega) \right\} \quad (6)$$

and  $\vec{v}_{ext}$  is evaluated on the surface of the body (and can be approximated as on the undeformed body) at the direction of the spatial angle  $\Omega$ .

It is convenient to write the correlation function of the external velocity field in the momentum space. This is so because the random velocity field is transversal when the fluid is incompressible. Consequently, in real space, the flow must always be correlated in a very complex way. On the other hand, in the momentum space we can simply use a projection operator on the transversal part of a general field:

$$\tilde{v}_{ext_i}(\vec{q}) \equiv \sum_j \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) u_j(\vec{q}), \quad (7)$$

where  $\vec{u}$  is a general vector field and the bracketed term is the projection operator that removes the longitudinal part of  $\vec{u}$ , and therefore yields a general transversal velocity field  $\tilde{v}_{ext}$ .

Next, the correlations of the external velocity are easily expressed using the correlations of the general field  $\vec{u}$ . We are interested in cases where the system is isotropic, homogeneous

and stationary. In these cases, the general field must obey:

$$\langle u_l(\vec{q}, t_1) u_m(\vec{p}, t_2) \rangle = \delta_{lm} \delta(\vec{q} + \vec{p}) \phi(q, t_2 - t_1), \quad (8)$$

where  $\delta_{lm}$  is the Kronecker delta,  $\delta()$  is the Dirac delta function and  $\phi()$  is a general function of  $q$  and  $\Delta t$ . In addition we have assumed that

$$\langle u_l(\vec{q}, t) \rangle = 0. \quad (9)$$

In the rest of this section we consider a frequently used family of random flows in which the external velocity is uncorrelated in time.

$$\phi(q, t_2 - t_1) = \tilde{\phi}(q) \delta(t_2 - t_1) \quad (10)$$

Equation (10) is a reasonable representative of systems in which the smallest time scale is that of the random velocity. In these systems we can replace the exact correlation details by the effective delta function by integrating on time.

Using Fourier transform and the definition of the correlations we can calculate the correlation of the velocity at two places on the drop, that are characterized by the directions  $\hat{r}$  and  $\hat{r}'$ . The calculation yields

$$\langle v_{ext}^i(\hat{r}, t_1) v_{ext}^j(\hat{r}', t_2) \rangle = \int d\vec{q} e^{-i\vec{q} \cdot (\vec{r} - \vec{r}')} \left[ \delta_{ij} - \frac{q_i q_j}{q^2} \right] \phi(|\vec{q}|) \delta(t_2 - t_1), \quad (11)$$

where  $\vec{v}_{ext}(\hat{r}, t)$  is the velocity at time  $t$  at place  $\vec{r}$  on the surface and  $\vec{r} \equiv \vec{r}_0(t) + R\hat{r}$ . The average and correlations of  $Q_{l,m}$  follow easily from the previous equations. The average of  $Q_{lm}$  is zero,

$$\langle Q_{lm} \rangle = 0. \quad (12)$$

The term  $\hat{\rho} \cdot \dot{\vec{r}}_0$ , in eq. (6), does not contribute to any component of  $Q_{lm}$  except for those with  $l = 1$ . In addition, the center,  $\vec{r}_0$ , have been chosen to be the point around which the three deformation coefficients,  $f_{lm}$ , with  $l = 1$  are equal to zero. Therefore  $Q_{1m}$  is zero, and for all other choices  $\dot{\vec{r}}_0$  can be dropped out of  $Q_{lm}$ .

The external velocity on the surface is approximated, in the first order of deformation,

as the external velocity on the undeformed sphere [1]. Strait forward calculation of  $Q_{l,m}$  correlations using its definition, eq. (6), and the velocity correlations, eqs. (7-10), yields:

$$\langle Q_{lm}(t_1)Q_{l'm'}(t_2) \rangle = \tilde{Q}_{lml'm'}\delta(t_2 - t_1), \quad (13)$$

where

$$\tilde{Q}_{lml'm'} \equiv \frac{1}{R^2} \int d\Omega \int d\Omega' \int d^3q Y_{lm}^*(\Omega) Y_{l'm'}^*(\Omega') \sum_{i,j=x,y,z} \left[ \hat{r}_i \hat{r}'_j e^{-i\vec{q} \cdot (\hat{r} - \hat{r}') R} \left[ \delta_{ij} - \frac{q_i q_j}{q^2} \right] \phi(|q|) \right]. \quad (14)$$

Finally, the correlations of the deformation coefficients,  $f_{lm}$ , are obtained, using Eq. (5) and the correlations of  $Q_{lm}$ , by direct integration

$$\langle f_{lm}(t)f_{l'm'}(t + \Delta t) \rangle_{t \rightarrow \infty} = \lim_{t \rightarrow \infty} \int_0^t dt_1 \int_0^{t+\Delta t} dt_2 \left[ \langle Q_{lm}(t_1)Q_{l'm'}(t_2) \rangle e^{\lambda_l(t_1-t) + \lambda_{l'}(t_2-t-\Delta t)} \right], \quad (15)$$

where the limit  $t \rightarrow \infty$  is taken to avoid the initial conditions that are imposed on a specific realization.

The last equation combined with Eqs. (14) and (13) yields

$$\langle f_{lm}(t)f_{l'm'}(t + \Delta t) \rangle_{t \rightarrow \infty} = \tilde{Q}_{lml'm'} \frac{e^{-\lambda_{l'}|\Delta t|}}{\lambda_l + \lambda_{l'}}. \quad (16)$$

Since the system is invariant to time reversal it is obvious that there is no preference of  $\lambda_{l'}$  over  $\lambda_l$  in the exponent, and therefore the result must equal zero for  $l \neq l'$ . In fact, only for the terms in which  $l = l'$  and  $m = -m'$   $\langle Q_{lm}(0)Q_{l'm'}(0) \rangle \neq 0$ . This follows from the fact that in the first order  $Q_{lm}$  depends only on the component of the external velocity that is normal to the surface. Therefore, its correlation function for two places at the same time must depend only on the angle between their direction, and can be expanded as  $\sum_l A_l P_l(\cos(\theta))$  where  $P_l$  is the Legendre polynomial. This can be turned into a sum of spherical harmonics using the partial waves expansion

$$P_l(\cos(\theta)) = \frac{4\pi}{2l+1} \sum_{m=-l}^l (-1)^m Y_{lm}(\Omega) Y_{l-m}(\Omega') \quad (17)$$

Now, it is obvious that the correlations of the deformation function,  $f(\Omega)$ , must have the same form

$$\langle f(\Omega)f(\Omega') \rangle = \sum_l B_l \frac{4\pi}{2l+1} \sum_{m=-l}^l (-1)^m Y_{lm}(\Omega) Y_{l-m}(\Omega'). \quad (18)$$

Furthermore,

$$\langle f_{lm} f_{l'm'} \rangle = \int \langle f(\Omega) f(\Omega') \rangle Y_{lm}^*(\Omega) Y_{l'm'}^*(\Omega') d\Omega d\Omega'. \quad (19)$$

Therefore, the expansion is composed of terms in which  $l = l'$  and  $m = -m'$  only, as stated above.

#### IV. GENERAL NOISE

In many cases white noise correlations are not sufficient to describe what really happens in the liquid. Especially if the correlation time is of the order of other time parameters. Such is the case of a system of many droplets inside a host fluid. The random flow, a droplet is subjected to, results from the random motion and deformation of other droplets that pass nearby. It is obvious that in this case the approximation of the flow to be uncorrelated in time is not justified. Hence, we need to generalize the theory.

Suppose that  $\phi(q, \Delta t)$  is a general function of  $q$  and the time differences  $\Delta t$ . The correlations of the external velocity are now extended in time. In order to calculate averages on the droplet in different times we must now consider also the movement of the droplet. The definition of  $Q_{lm}$  (6) implies that the correlations of the normal component of the external velocity field on the surface of the body are

$$\langle Q_{lm}(t_1) Q_{l'm'}(t_2) \rangle = \frac{1}{R^2} \int d\Omega \int d\Omega' Y_{lm}^*(\Omega) Y_{l'm'}^*(\Omega') \sum_{i,j=x,y,z} \left[ \hat{r}_i \hat{r}'_j \left\langle v_{ext}^i(\hat{r}, t_1) v_{ext}^j(\hat{r}', t_2) \right\rangle \right]. \quad (20)$$



The correlation of the velocity at two points on the droplet, located in the directions  $\hat{r}$  and  $\hat{r}'$  and measured at different moments, obviously depends on the displacement of the center,  $\Delta\vec{r}_0$ . We calculate first the correlation for a general displacement of the center and then, average the result according to the probability of finding the center at each point. To do this, we first express the velocity correlations in means of the Fourier transform of the velocity, use equations (7),(8) and obtain

$$\langle v_{ext}^i(\hat{r}, t_1) v_{ext}^j(\hat{r}', t_2) \rangle = \int P(\Delta\vec{r}_0) d(\Delta\vec{r}_0) \int d^3q e^{-i\vec{q} \cdot (\Delta\vec{r}_0 + R(\hat{r} - \hat{r}'))} [\delta_{ij} - \frac{q_i q_j}{q^2}] \phi(q, \Delta t), \quad (21)$$

where  $\Delta\vec{r}_0 = \vec{r}_0(t_1) - \vec{r}_0(t_2)$ ,  $\Delta t = t_2 - t_1$ ,  $P(\Delta r_0)$  is the probability that the center will be displaced by  $\Delta\vec{r}_0$  in the period of  $\Delta t$  and the summation over  $d(\Delta\vec{r}_0)$  is taken on all the possible configurations.

It is obvious now that averaging over the center displacements will effect only the term  $-i\vec{q} \cdot \Delta\vec{r}_0$  in the exponent since this is the only term that depends on  $\Delta\vec{r}_0$ . Consequently,

$$\langle v_{ext}^i(\hat{r}, t_1) v_{ext}^j(\hat{r}', t_2) \rangle = \int d^3q \langle e^{-i\vec{q} \cdot \Delta\vec{r}_0} \rangle e^{-i\vec{q} \cdot (\hat{r} - \hat{r}')R} (\delta_{ij} - \frac{q_i q_j}{q^2}) \phi(q, \Delta t), \quad (22)$$

Assuming Gaussian distribution of the displacements of the center,

$$\langle e^{-i\vec{q} \cdot \Delta\vec{r}_0} \rangle = e^{-\frac{q^2}{6} \langle (\Delta\vec{r}_0)^2 \rangle}. \quad (23)$$

In a previous paper [1], we considered the Mean Squared Displacement (MSD) of the center of a deformable body in a flow that is correlated in a general way. We found that the MSD in a period of time  $\Delta t$  is given by:

$$F(\Delta t) = 16\pi \int_0^{\Delta t} dt' \int_0^\infty q^2 dq e^{-\frac{q^2}{6} F(t')} \phi(q, t') (j_0(qR) + j_2(qR))^2 (\Delta t - t'), \quad (24)$$

where  $F(\Delta t) \equiv \langle (\Delta\vec{r}_0)^2 \rangle$ . Therefore the correlation of the external velocity at two points on the surface, characterized by the directions  $\hat{r}$  and  $\hat{r}'$ , measured at two different times with time gap of  $\Delta t$ , is given by

$$\langle v_{ext}^i(\hat{r}, t_1) v_{ext}^j(\hat{r}', t_2) \rangle = \int d^3q e^{-\frac{q^2}{6} F(\Delta t)} e^{-i\vec{q} \cdot (\hat{r} - \hat{r}')R} (\delta_{ij} - \frac{q_i q_j}{q^2}) \phi(q, \Delta t), \quad (25)$$

Equations (24) and (25) enable us to calculate the correlations of  $Q_{l,m}$  (eq. (20)). The correlations of the deformation coefficients can be obtained from their basic equation (5) using the correlations of  $Q_{l,m}$ ,

$$\langle f_{l,m}(t)f_{l',m'}(t+\Delta t) \rangle_{t \rightarrow \infty} = \int_0^t dt' \int_0^{t+\Delta t} dt'' \frac{\langle Q_{l,m}(t')Q_{l',m'}(t'') \rangle \exp(\lambda_l t' + \lambda_{l'} t'')}{\exp(\lambda_l t + \lambda_{l'}(t+\Delta t))}. \quad (26)$$

Equations (20), (24), (25) and (26) form the calculation method for the correlations of the deformation coefficients. The method presented here may be hard to implement to numerical calculations because of the many dimensions integration. In the appendix we describe an algorithm that reduces the integrations to be one dimensional, thus making the computation task easier. This algorithm was used to obtain the results presented in the following section. At same times,  $\Delta t = 0$ , the correlations above play an important role in the shape distribution of the drop. They hold all the information needed to determine the distribution of the  $f_{l,m}$ 's. As was explained in the previous section, same time correlation (STC) are obviously non-zero only for  $l = l'$  and  $m = -m'$ . In addition we consider cases where  $\langle f_{l,m} \rangle = 0$ . Thus  $\langle f_{l,m}(t)f_{l,-m}(t) \rangle$  give the variance of the deformation coefficient  $f_{l,m}$ , and since  $f_{l,m}$  with different  $l, m$  are uncorrelated we can use this variance to create typical shapes of drops (Fig. 1).

Equation (26) can be simplify a little in the case of STC

$$\langle f_{l,m}(t)f_{l',m'}(t) \rangle_{t \rightarrow \infty} = \int_0^\infty dt' \langle Q_{l,m}(0)Q_{l',m'}(t') \rangle \frac{\exp(-\lambda_l t') + \exp(-\lambda_{l'} t')}{\lambda_l + \lambda_{l'}}. \quad (27)$$

## V. THE CORRELATION FUNCTION

We wanted the method presented above to be applicable to different random external flows. Therefore, we have built the calculation method in such a way that the correlation function  $\phi(q, t)$  must be given by hand. In some cases  $\phi(q, t)$  can be calculated theoretically. In others, to obtain  $\phi(q, t)$  for a given system one can make an experiment. Exclude the deformable body from the system; measure the correlations in the velocity in different points at different times and deduce the correlation function  $\phi(q, t)$ . To confirm the results we

present here just insert back the deformable body, measure the correlations and compare with the theoretical prediction. We can also try to use our method to obtain the correlation function itself. Of particular interest is the response of a system of deformable bodies to an external flow. The deformation of each body will induce an additional velocity field that will influence the others. Hence, the correlations in the velocity, each body is subjected to, can be treated in a self consistent manner, using the deformations as a source to the external velocity field as well as the field's product.

In the rest of this section we will analyze examples of random flows.

Consider first the special case of thermal agitation. In a previous paper [1] we have shown that the correlation function for the external velocity due to thermal agitation have the form  $\phi(q, t) = \frac{K_B T}{(2\pi)^3 \eta} \frac{\delta(t)}{q^2}$ , where  $\eta$  is the viscosity of the fluid (notice that the correlations are long ranged due to the incompressibility of the fluid). Using this form with eq. (16) the STC of the deformation coefficient,  $f_{l,m}$ , where  $l > 1$  is given by

$$\langle f_{l,m}(t) f_{l,-m}(t) \rangle_{t \rightarrow \infty} = \frac{K_B T}{R^2 \lambda} \frac{1}{(l+2)(l-1)} \quad (28)$$

Schwartz and Edwards [5] considered the special case of deformable a body in equilibrium at temperature  $T$ , using the Kirkwood equation. They found identical correlations in the deformations (using  $X_{l,m} = R f_{l,m}$ ). Their derivation, however, has been tailored for thermal agitation and cannot be expanded to take into account any other correlations in the host fluid.

The shape of the deformable body is described by  $r = R(1 + f(\Omega))$ . Hence, eq. (28) implies that the magnitude of the deformation,  $R f_{l,m}$ , does not depend on the size of the droplet, i.e. the deformations should be as of an infinite lamellar surface.

Thermal agitation belongs to a class of systems in which the decay time, which is the time that takes the external velocity field to loose memory and become uncorrelated, is the shortest time scale (Although other time scales  $\tau_l = \frac{1}{\lambda_l}$  tend to zero for  $l \rightarrow \infty$  we can use this form to any finite set of spherical harmonics. In addition we must remember that

$l$  is limited by the fact that features on the sphere must be larger than the inter-atomic distance for the continuum description to hold). In these cases we can approximate the actual velocity field to be totally uncorrelated in time.

$$\phi(q, t) = C\tilde{\phi}(q)\delta(t). \quad (29)$$

There are two possible scenarios: either  $\tilde{\phi}(q)$  decay with a length scale,  $\xi$  (at least one), or it is long-ranged. We limit our discussion to one power and one length scale and the generalization is trivial. In the first case  $\tilde{\phi}(q) = q^{-\alpha}\tilde{\phi}(\xi q)$ , where  $\tilde{\phi}$  has a cut off at  $\xi q = 1$ . There are two dimensionless parameters for the STC,  $\mu_1 = \frac{R}{\xi}$  and  $\mu_2 = \frac{C}{R^{5-\alpha}\lambda_l}$ . In the case of a body characterized by surface tension energy,  $\lambda_l \propto \frac{\lambda}{\eta R}$  where  $\lambda$  is the surface tension and  $\eta$  is the viscosity,

$$\langle f_{l,m}(t)f_{l',m'}(t) \rangle_{t \rightarrow \infty} = F_{l,m,l',m'} \left( \frac{C\eta}{R^{4-\alpha}\lambda}; \frac{R}{\xi} \right). \quad (30)$$

Fig. 3 and 4 depict the STC dependence on  $l$  for two correlation functions and different parameters values where we changed  $\mu_1$  and  $\mu_2$  by keeping  $\xi = 1$  and  $\frac{C\eta}{\lambda} = 1$  and changing  $R$ . As can be seen, there are two possibilities: either the STC with  $l = 2$  dominates the curve or there is a maximum at  $l_0 \approx \frac{R}{\xi}$ . It is easy to see that the decay, for  $l \gg l_0$  is exponential. This suggests that there is a cutoff on the deformation coefficients at which the expansion can be terminated and therefore that the expansion we use here will be useful for systems in which  $\mu_1$  is small.

Fig 1 depicts various shapes of a body characterized by surface tension under a random flow, given by  $\phi(q, t) = C\delta(\xi q - 1)\delta(t)$ . As can be seen, the surface of the body develops bumps. It is obvious that the size of these surface features depends on the ratio  $\mu_1 = R/\xi$ , (Fig. 1). As  $\gamma$  increases, different surface elements become less and less correlated. Therefore, we expect to see features of smaller size (which correspond, clearly, to spherical harmonics of higher order). The smallest features correspond to deformation coefficients with  $l \simeq \mu_1$  (or  $l = 2$  if  $\mu_1 \leq 2$ ).

In the second case the parameter  $\mu_1$  can be dropped out and we are left only with  $\mu_2$ , as in

the case of thermal agitation.

The correlation of the deformation coefficients for different times  $\langle f_{l,m}(t)f_{l',m'}(t + \Delta t) \rangle_{t \rightarrow \infty}$  depend in addition on the dimensional parameter  $\mu_3 = \lambda_l \Delta t$  exponentially as written in eq. 16.

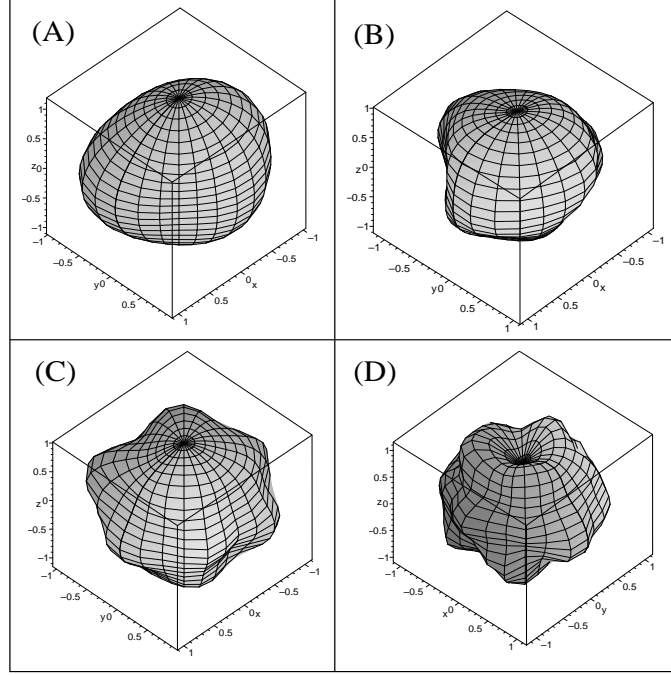


FIG. 1. A typical realization of a deformable body subjected to a random flow of the form  $\phi(q) = C\delta(q\xi - 1)\delta(t)$  with (A)  $R = 2\xi$ , (B)  $R = 4\xi$ , (C)  $R = 6\xi$  and (D)  $R = 8\xi$ .

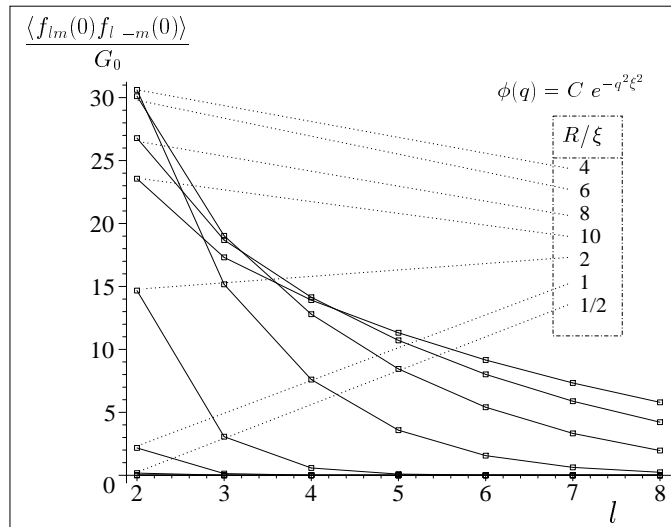


FIG. 2. The variance of the deformation coefficients,  $f_{lm}$ , as a function of  $l$ , for a typical decaying external noise :  $\phi(q) = Ce^{-q^2\xi^2}$ . ( $G_0 \equiv \mu_2$ ).

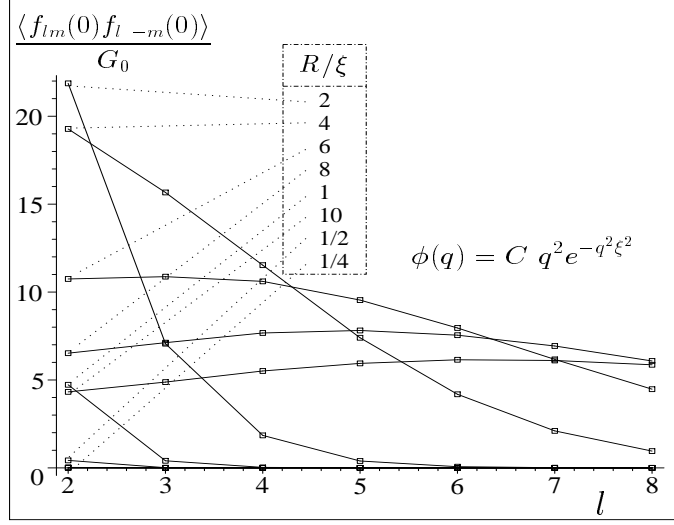
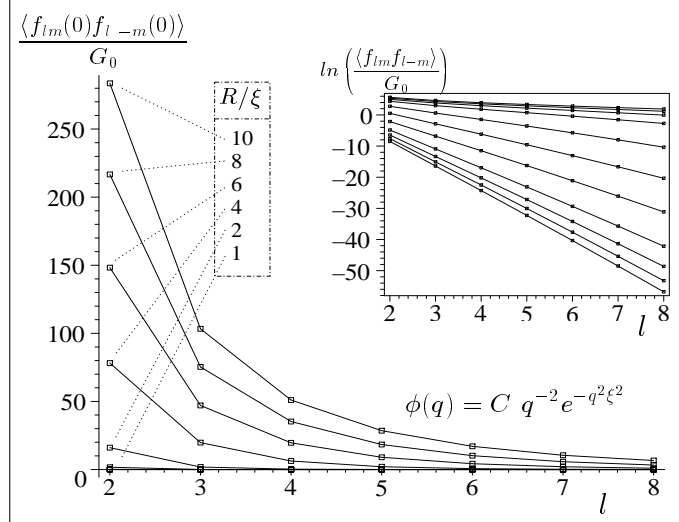


FIG. 3. The variance of the deformation coefficients,  $f_{lm}$ , as a function of  $l$ , for  $\phi(q) = C q^2 e^{-q^2 \xi^2}$ . ( $G_0 \equiv \mu_2$ ).

non-



separable

FIG. 4. The variance of the deformation coefficients,  $f_{lm}$ , as a function of  $l$ .  $\phi(q) = C q^{-2} e^{-q^2 \xi^2}$ . For  $R \gg \xi$  the curves coincide with the curves for thermal agitation. ( $G_0 \equiv \mu_2$ ).

Other classes of random flows include separable correlation functions in which there is a decay length scale,  $\xi$ , and decay time scale  $\tau$ :  $\phi(q, t) = C q^{-\alpha} \tilde{\phi}(\xi q) T(t/\tau)$ , where  $\langle f_{l,m}(t) f_{l',m'}(t) \rangle_{t \rightarrow \infty} = F_{l,m,l',m'} \left( \frac{C \tau^2}{R^{5-\alpha}}; \frac{R}{\xi}; \lambda_l \tau \right)$ , and flows of the form  $\phi(q, t) = C q^{-\alpha} \tilde{\phi}(\Gamma q t^\beta)$  for which analog dimensional parameters can be written.

## VI. SUMMARY

We have built a method to calculate the correlations of the deformation coefficients  $f_{l,m}$  that correspond to the decomposition of the shape into spherical harmonics. We did it in two stages. The first, for external velocity fields that are uncorrelated in time: eqs. (14), (16). The second, for a general external velocity field that is correlated both in space and time: eqs. (20), (24), (25) and (26). The algorithm was based on the assumption that the correlation of the velocity field in an identical system without the body are known. We discussed these correlations, used the results to build in a rigorous way typical surface shapes and considered the special case of thermal agitation. In addition, we pointed out that from our numerical results there is a parameter  $\mu_1 = R/\xi$  where deformation coefficients with  $l > \mu_1$  seem to decay exponentially and therefore are essentially unimportant. This suggests that working with spherical harmonics to investigate systems of deformable bodies is extremely useful for cases where  $\mu_1$  is small.

## APPENDIX A: ALGORITHM FOR THE CALCULATION OF THE DEFORMATION CORRELATIONS

The velocity correlations involve a three dimensional integration, and Eq. (14) uses an additional four dimensional integration. As we can see, the method presented above is very hard to use. Therefore we must find a way to lower the dimensionality of the integrals. the following algorithm illustrates a method to do so using the addition theorem (or partial waves expansion). The result is a finite expression which is composed of sum of terms where each term involves only one dimensional integration. Unfortunately, although finite, this sum is too long to be presented here.

The partial waves expansion is given by

$$e^{-i\vec{q}\cdot(R\hat{\rho})} = \sum_{l=0}^{\infty} \sum_{m=-l}^l (-i)^l 4\pi j_l(qR) Y_{lm}^*(\Omega_q) Y_{lm}(\Omega), \quad (\text{A1})$$

where  $\Omega_q$  is the solid angle in the  $\vec{q}$  direction and  $j_l$  is the spherical Bessel function. Using the partial waves expansion with Eq. (11) yields

$$\langle v_{ext}^i(\vec{r}(t)) v_{ext}^j(\vec{r}(t + \Delta t)) \rangle = \sum_{l_1, l_2, m_1, m_2} A_{l_1 m_1 l_2 m_2 ij} Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}^*(\Omega'), \quad (\text{A2})$$

where

$$A_{l_1 m_1 l_2 m_2 ij} \equiv (-i)^{l_1} i^{l_2} (4\pi)^2 \int d^3 q e^{-\frac{q^2}{6} F(\Delta t)} j_{l_1}(qR) j_{l_2}(qR) \cdot Y_{l_1 m_1}^*(\Omega_q) Y_{l_2 m_2}(\Omega_q) \left[ \delta_{ij} - \frac{q_i q_j}{q^2} \right] \phi(|\vec{q}|, \Delta t) \quad (\text{A3})$$

Each coefficient  $A_{l m l' m' ij}$  divides easily into two parts, one deals with the angular integration and the other with the radial integration. The angular part is easily integrated, by the use of the Clebsch-Gordan coefficients, to give an algebraic expression. Therefore to evaluate  $A_{l m l' m' ij}$  one needs to calculate the radial integration which is one dimensional.

We use Eq. (A2) and obtain a new equation for the  $Q_{lm}$  correlations.

$$\langle Q_{lm}(0) Q_{l'm'}(\Delta t) \rangle \equiv \frac{1}{R^2} \int d\Omega \int d\Omega' Y_{lm}^*(\Omega) Y_{l'm'}(\Omega') \sum_{i,j=x,y,z} \left[ \hat{r}_i \hat{r}'_j \sum_{l_1, m_1, l_2, m_2} A_{l_1 m_1 l_2 m_2 ij} Y_{l_2 m_2}^*(\Omega') Y_{l_1 m_1}(\Omega) \right] \quad (\text{A4})$$

The integrations over  $\Omega$  and  $\Omega'$  can be easily calculated once we take into consideration that unit vectors can be decomposed into spherical harmonics with  $l = 0, 1$ . The integration yields a finite sum of terms. Each term is composed by a known number,  $\zeta$ , multiplied by the corresponding  $A_{l_1, m_1, l_2, m_2, ij}$  that involves only one dimensional integration. Hence,

$$\langle Q_{lm}(0) Q_{l'm'}(\Delta t) \rangle \equiv \sum_{l_1, l_2, m_1, m_2, i, j} \zeta_{l_1, l_2, m_1, m_2, i, j}(l, l', m, m') \int q^2 dq e^{-\frac{q^2}{6} F(\Delta t)} j_l(qR) j_{l'}(qR) \phi(|\vec{q}|, \Delta t) \quad (\text{A5})$$

In this way we formulated a method for the calculation of the correlations. The explicit expression is extremely long and therefore is not presented in this paper.



- [1] M.Schwartz and G. Frenkel, cond-mat/0110075 (2001)
- [2] G. Frenkel and M. Schwartz, Physica A 298 (2001) 278
- [3] M. Schwartz and S.F. Edwards, Physica A 153 (1988) 355
- [4] S.F. Edwards and M. Schwartz, Physica A 167 (1990) 595
- [5] S.F. Edwards and M. Schwartz, Physica A 178 (1991) 236
- [6] Y. Navot, Phys. of fluids 11 no. 5 (1999) 990
- [7] H.J. Deuling and W. Helfrich, Journal-de-Physique 37 no. 11 (1976) 1335
- [8] V. Lisy, B. Brutovsky and A.V. Zatovsky, Phys. Rev. E 58 no. 6 (1998) 7598